



8.5 The null space of a linear map





Introduction

- In this topic, we will
 - Define the null space of a linear map
 - Consider some consequences
 - Look at finite-dimensional maps
 - See the relationship between the range and the null space
 - Consider the advance and delay operators





Describing linear maps

- Recall that we want to describe the useful characteristics of a linear map
 - This is not information we can get by looking at, for example, the individual numbers in a given matrix
 - The first two such features are to describe the *range* and *null space* of that linear map
 - The last topic covered the range,
now we will describe the null space





The null space

- Given a linear map $A: \mathcal{U} \rightarrow \mathcal{V}$, the null space is the collection of all vectors in $\mathcal{U}$ that map onto $\mathbf{0}_{\mathcal{V}}$.
- That is, $\text{null}(A)$ equals all vectors $\mathbf{u}$ where $A\mathbf{u} = \mathbf{0}_{\mathcal{V}}$
 - You will note that $\text{null}(A) \subseteq \mathcal{U}$
 - Also, $\text{null}(A)$ always has at least one element: $\mathbf{0}_{\mathcal{U}}$





The null space is a subspace

Theorem

If $A: \mathcal{U} \rightarrow \mathcal{V}$ is a linear map,
the null space of A is a subspace of $\mathcal{U}$

Proof:

If $\mathbf{u}_1, \mathbf{u}_2 \in \text{null}(A)$, then we must show $\alpha \mathbf{u}_1 + \beta \mathbf{u}_2 \in \text{null}(A)$

$$\begin{aligned} \text{But because } A \text{ is linear, } A(\alpha \mathbf{u}_1 + \beta \mathbf{u}_2) &= \alpha A\mathbf{u}_1 + \beta A\mathbf{u}_2 \\ &= \alpha \mathbf{0}_{\mathcal{V}} + \beta \mathbf{0}_{\mathcal{V}} = \mathbf{0}_{\mathcal{V}} \end{aligned}$$

Thus, $\alpha \mathbf{u}_1 + \beta \mathbf{u}_2 \in \text{null}(A)$

Thus, the null space is closed under vector addition
and scalar multiplication, so it is a subspace. ■





Solutions to $A\mathbf{u} = \mathbf{v}$

Theorem

If $A: \mathcal{U} \rightarrow \mathcal{V}$ is a linear map, and $A\mathbf{u} = \mathbf{v}$ and $\mathbf{u}_0 \in \text{null}(A)$,
then $A(\mathbf{u} + \alpha \mathbf{u}_0) = \mathbf{v}$ for all $\alpha \in \mathbf{F}$

Proof:

$$\begin{aligned} \text{Because } A \text{ is linear, we have } A(\mathbf{u} + \alpha \mathbf{u}_0) &= A\mathbf{u} + \alpha A\mathbf{u}_0 \\ &= \mathbf{v} + \alpha \mathbf{0}_{\mathcal{V}} \\ &= \mathbf{v} + \mathbf{0}_{\mathcal{V}} = \mathbf{v} \quad \blacksquare \end{aligned}$$





The difference of two solutions to $A\mathbf{u} = \mathbf{v}$

Theorem

If $A: \mathcal{U} \rightarrow \mathcal{V}$ is a linear map, and $A\mathbf{u}_1 = A\mathbf{u}_2 = \mathbf{v}$,
then $\mathbf{u}_1 - \mathbf{u}_2 \in \text{null}(A)$.

Proof:

$$\begin{aligned} \text{Because } A \text{ is linear, we have } A(\mathbf{u}_1 - \mathbf{u}_2) &= A\mathbf{u}_1 - A\mathbf{u}_2 \\ &= \mathbf{v} - \mathbf{v} \\ &= \mathbf{0}_{\mathcal{V}} \blacksquare \end{aligned}$$





Definition: one-to-one

Definition

If $A: \mathcal{U} \rightarrow \mathcal{V}$ and if for each $\mathbf{u} \in \mathcal{U}$, the value $A\mathbf{u}$ is unique, we will say that A is one-to-one.

That is, each vector $\mathbf{v} \in \text{range}(A)$ has a unique pre-image

It is also said that A is *injective*.





Definition: one-to-one

- What is “one-to-one”?
 - A function is one-to-one if different t values evaluate to different y values
 - Any linear polynomial $at + b$ with a non-zero slope is one-to-one
 - The exponential function e^t is one-to-one
 - Any differentiable function $y(t)$ such that $\frac{d}{dt}y(t) > 0$ for most t and $\frac{d}{dt}y(t) = 0$ for only isolated points is one-to-one
 - No even polynomial with real coefficients and of a real variable is one-to-one
 - For $t > 0$, $\ln(t)$ and $\sqrt{t}$ are one-to-one
- Challenge: prove that every polynomial of degree greater than one with complex coefficients is never one-to-one





When is a linear map one-to-one

Theorem

If $A: \mathcal{U} \rightarrow \mathcal{V}$ is a linear map, then A is one-to-one if and only if the null space is just the zero vector; i.e., $\text{null}(A) = \{\mathbf{0}_{\mathcal{U}}\}$.

Proof:

If A is one-to-one, then only one vector in $\mathcal{U}$ can map onto $\mathbf{0}_{\mathcal{V}}$, and since $A\mathbf{0}_{\mathcal{U}} = \mathbf{0}_{\mathcal{V}}$, it follows $\text{null}(A) = \{\mathbf{0}_{\mathcal{U}}\}$.

If $\text{null}(A) = \{\mathbf{0}_{\mathcal{U}}\}$, suppose $A\mathbf{u}_1 = A\mathbf{u}_2$.

From a previous theorem, we have $\mathbf{u}_1 - \mathbf{u}_2 \in \text{null}(A)$.

But therefore, $\mathbf{u}_1 - \mathbf{u}_2 = \mathbf{0}_{\mathcal{U}}$, so $\mathbf{u}_1 = \mathbf{u}_2$.

Therefore, A is one-to-one. ■





Range of finite-dimensional linear mappings

- If $A: \mathbf{F}^n \rightarrow \mathbf{F}^m$, the null space is found by solving the homogenous system of linear equations
- For example,

$$A = \left(\begin{array}{ccccc|c} 1 & 2 & -1 & -3 & -2 & 0 \\ 3 & 8 & 2 & -11 & -2 & 0 \\ 4 & 14 & 11 & -18 & 7 & 0 \\ -2 & -8 & -8 & 10 & -10 & 0 \end{array} \right) \sim \left(\begin{array}{ccccc|c} 1 & 2 & -1 & -3 & -2 & 0 \\ 0 & 2 & 5 & -2 & 4 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

- We see that $3x_5 = 0$, so $x_5 = 0$
- Also, x_3 and x_4 are free variables
- Thus, $2x_2 + 5x_3 - 2x_4 = 0$,
so $x_2 = -2.5x_3 + x_4$
- Thus, $x_1 + 2x_2 - x_3 - 3x_4 = 0$,
so $x_1 = 6x_3 + x_4$

$$\mathbf{x} = \begin{pmatrix} 6x_3 + x_4 \\ -2.5x_3 + x_4 \\ x_3 \\ x_4 \\ 0 \end{pmatrix} = x_3 \begin{pmatrix} 6 \\ -2.5 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$





The range and the null space

Theorem

If $A: \mathbf{F}^n \rightarrow \mathbf{F}^m$ is a linear map, the dimension of the null space plus the dimension of the range equals n .

- That is, $\dim(\text{null}(A)) + \dim(\text{range}(A)) = n$

Proof:

Convert the matrix to row-echelon form with operations:

- Each column associated with a leading non-zero coefficient contributes one vector to the basis of the range,

thus the dimension of the range is the number of columns with leading non-zero entries.

- Each column associated with a free variable contributes one vector to the basis of the null space,

thus, the dimension of the null space is the number of columns without leading non-zero entries.

There are n columns, so these two add up to n . ■





When a matrix can never be onto

Theorem

If $A: \mathbf{F}^n \rightarrow \mathbf{F}^m$ is a linear map, and $n < m$,
then the dimension of the range must be less than m .

Proof:

The maximum $\text{rank}(A) = n$,
so the maximum dimension of $\text{range}(A)$ is n

Thus $\dim(\text{range}(A)) \leq n < m$. ■

- Thus, if $n < m$, the linear map is never onto.





When a matrix can never be one-to-one

Theorem

If $A: \mathbb{F}^n \rightarrow \mathbb{F}^m$ is a linear map, and $n > m$,
then the dimension of the null space must be greater
than or equal to $n - m$

Proof:

The maximum $\text{rank}(A) = m$,
so the maximum dimension of $\text{range}(A)$ is m

$$\dim(\text{range}(A)) \leq m$$

$$-\dim(\text{range}(A)) \geq -m$$

But $\dim(\text{range}(A)) + \dim(\text{null}(A)) = n$

Adding these last two, we get $\dim(\text{null}(A)) \geq n - m$. ■

- Thus, if $n > m$, the linear map is never one-to-one.





When can a matrix be one-to-one and onto?

Theorem

If $A: \mathbf{F}^n \rightarrow \mathbf{F}^m$ is a linear map,
then for A to be one-to-one and onto, $n = m$

Proof:

For a linear map to be onto, $n \geq m$.

For a linear map to be one-to-one, $n \leq m$.

Consequently, for such a matrix to be one-to-one and onto,
it follows that $n = m$. ■

- Important: Not all linear operators $A: \mathbf{F}^n \rightarrow \mathbf{F}^n$ are one-to-one and onto, but for a linear map to be one-to-one and onto, it must be a linear operator





When is a matrix is one-to-one and onto?

Theorem

If $A: \mathbb{F}^n \rightarrow \mathbb{F}^n$ is a linear operator,
then A is either both one-to-one and onto, or neither.

Proof:

If A is onto, then $\dim(\text{range}(A)) = n$.

Thus, as $\dim(\text{range}(A)) + \dim(\text{null}(A)) = n$,
it follows that $\dim(\text{null}(A)) = 0$, so A is one-to-one.

If A is not onto, then $\dim(\text{range}(A)) < n$.

Thus, as $\dim(\text{range}(A)) + \dim(\text{null}(A)) = n$,

adding $-\dim(\text{range}(A)) > -n$

Thus, $\dim(\text{null}(A)) > 0$, so A is not one-to-one. ■





Applications of these theorems

- Given

$$A = \begin{pmatrix} 2.3 & -1.2 & -0.5 & 2.5 & 3.1 & -1.3 \\ 2.1 & 4.5 & 3.2 & -1.3 & -2.4 & 0.9 \\ 4.1 & 1.3 & 1.2 & -1.8 & 1.3 & 0.4 \\ -1.3 & -1.5 & -1.8 & 1.0 & -1.3 & -2.4 \end{pmatrix}$$

- We note $A: \mathbf{R}^6 \rightarrow \mathbf{R}^4$
 - Because $6 > 4$, this linear map cannot be one-to-one
 - The minimum dimension of the null space is two
 - If $\text{rank}(A) = 4$, this matrix is onto,
but if $\text{rank}(A) < 4$, this matrix is not onto





Applications of these theorems

- Given
$$B = \begin{pmatrix} 4.2 & -4.5 & -2.3 \\ 2.7 & 1.3 & 0.9 \\ -3.2 & 1.0 & 3.4 \\ 5.9 & -3.2 & -0.3 \\ 0.8 & -1.3 & -2.7 \\ -0.8 & 0.3 & 0.7 \\ 1.3 & 3.9 & -2.8 \end{pmatrix}$$
- We note $B: \mathbf{R}^3 \rightarrow \mathbf{R}^7$
 - Because $3 < 7$, this linear map cannot be onto
 - The maximum dimension of the range is three
 - If $\text{rank}(B) = 3$, this matrix is one-to-one, but if $\text{rank}(B) < 3$, this matrix is not one-to-one





Applications of these theorems

- Given

$$C = \begin{pmatrix} 4.8 & -4.2 & -4.4 & 18.6 \\ 8.0 & 8.0 & -9.0 & 11.0 \\ 5.6 & 2.9 & -3.0 & 10.8 \\ 4.0 & -3.2 & 2.3 & 16.1 \end{pmatrix} \sim \begin{pmatrix} 8 & 8 & -9 & 11 \\ 0 & -9 & 1 & 12 \\ 0 & 0 & 6 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

- We note $C: \mathbf{R}^4 \rightarrow \mathbf{R}^4$
 - Because, the domain and co-domain are equal, either C is both one-to-one and onto, or it is neither
 - If $\text{rank}(C) = 4$, this matrix is one-to-one and onto, but if $\text{rank}(C) < 4$, this matrix is neither one-to-one nor onto
 - We note that this matrix is both one-to-one and onto





Applications of these theorems

- Given

$$D = \begin{pmatrix} 7.2 & -9.0 & -2.4 & -20.8 \\ -8.0 & 10.0 & 2.0 & 11.0 \\ -7.2 & 9.0 & -4.2 & 0.9 \\ 7.2 & -9.0 & -5.4 & -21.3 \end{pmatrix} \sim \begin{pmatrix} -8 & 10 & 2 & 11 \\ 0 & 0 & -6 & -9 \\ 0 & 0 & 0 & -10 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- We note $D: \mathbf{R}^4 \rightarrow \mathbf{R}^4$
 - Because, the domain and co-domain are equal, either D is both one-to-one and onto, or it is neither
 - We can determine this by finding the rank
 - We note that this matrix is neither onto nor one-to-one





Operators on discrete-time signals

- Recall the delay and advance operators on discrete-time signals:
- If $x = (x[0], x[1], x[2], x[3], \dots)$,
 $Dx = (0, x[0], x[1], x[2], \dots)$ and the null space is just the zero signal
 $Ex = (x[1], x[2], x[3], x[4], \dots)$ and the null spaces is all signals with zero in all entries except for the first
- The rule we just saw, that a linear operator is either one-to-one and onto or neither, only applies to finite-dimensional vector spaces
 - D and E are both linear operators
on the vector space of discrete-time signals
 - D is one-to-one but not onto
 - E is onto but not one-to-one





Review

- Given a linear map $A: \mathbf{R}^n \rightarrow \mathbf{R}^m$,
by using Gaussian elimination and backward substitution
 - We can determine the range and find a basis for it
 - We can determine if it is onto
 - We can determine the null space and find a basis for it
 - We can determine if it is one-to-one
- These are all useful properties to understand about a given linear map or operator





Summary

- Following this topic, you now
 - Know the definition of the null space
 - Know that the null space of a linear map is always a subspace
 - Have become familiar with some additional results
 - Understand the relationship between matrices, the rank and the dimensions of the range and null space
 - Know how to determine a number of properties about a matrix





References

- [1] [https://en.wikipedia.org/wiki/Kernel_\(linear_algebra\)](https://en.wikipedia.org/wiki/Kernel_(linear_algebra))





Acknowledgments

None so far.





Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using Consolas. Mathematical equations are prepared in MathType by Design Science, Inc. Examples may be formulated and checked using Maple by Maplesoft, Inc.

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